ON A MIN-MAX CONJECTURE FOR REDUCIBLE DIGRAPHS

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ABSTRACT

A. Frank and A. Gyárfás (1976) have conjectured that in a reducible digraph D the maximum number of edge disjoint cycles equals the minimum number of edges intersecting all cycles of D. We prove this conjecture in the special case when D has at most two distinct dominators. The proof leads to a polynomial time algorithm for finding both the maximum set of cycles and minimum set of edges, in the considered case.

RESUMO

A. Frank e A. Gyárfás (1976) conjecturaram que em um digrafo redutível D o número máximo de ciclos disjuntos em arestas é igual ao número mínimo de arestas que interceptam todos os ciclos de D. Provamos essa conjectura no caso especial em que D possui no máximo dois denominadores distintos. A prova conduz a um algoritmo polinomial para encontrar tanto o conjunto máximo de ciclos quanto o conjunto mínimo de arestas, no caso considerado.

1. Introduction

A conjecture by A.Frank and A.Gyárfás [2] states that the maximum number of edge disjoint cycles of a reducible digraph D equals the minimum number of edges whose removal turns D acyclic. In the present paper we prove a special case of this conjecture. The proof is constructive and leads to a polynomial time algorithm for finding such a maximum set of cycles and minimum set of edges, in the considered case.

A flow digraph is a digraph D together with a vertex $s \in V(D)$, called root, such that every vertex of D is reachable from s. In particular, if every path from s to $v \in V(D)$ contains $w \in V(D)$ then w dominates v. A (fully) reducible digraph is a flow digraph D such that each cycle C of D contains some vertex w which dominates all the vertices of C. We call w a dominator of both C and D. See [5,7,9].

Let D be a general digraph. Denote by

α_V = set of vertex disjoint cycles of D

α_E = set of edge disjoint cycles of D

β_V = set of vertices intersecting all cycles of D

β_E = set of edges intersecting all cycles of D

Clearly, $\max|\alpha_V| \leq \min|\beta_V|$ and $\max|\alpha_E| \leq \min|\beta_E|$.

β_V and β_E are also known as feedback vertex and edge sets, respectively. Recall that the problems of finding the minimum cardinality sets α_V and α_E are both NP-hard [3,6].

Theorem 1 (Frank and Gyárfás [2]): If D is reducible then $\max|\alpha_V(D)|=\min|\beta_V(D)|$.

It follows that a minimum feedback vertex set of a reducible digraph can be found in polynomial time [2,4,8].

Conjecture [2]: If D is reducible then $\max|\alpha_E(D)|=\min|\beta_E(D)|$.

We prove this conjecture in the case when D has at most two distinct dominators.

2. The Proof

Throughout this section, D will always denote a reducible digraph.

Let $C \equiv v_1,\ldots,v_k,v_1$, $k>1$, be a cycle of D and v_1 the dominator of C. Then edge (v_k, v_1) is called a back edge of D.

Lemma 1: Each cycle of D contains exactly one back edge.

Proof: Let $v_1,\ldots,v_k,v_1$ be a cycle of D and v_1 its dominator. Then (v_k, v_1) is a back edge. Suppose that C contains another back edge (v_i,v_{i+1}), $i\neq k$. In this case, v_{i+1} is the dominator of another cycle C' which contains v_i. Let P be a path from s to v_1 followed by $v_1,\ldots,v_i$. Then P meets C' in a vertex which is not its dominator, a contradiction. That is, C has exactly one back edge▲

Let $\{r_1,\ldots,r_m\}$ be the set of dominators of D. Denote by D* the network obtained by the following construction:

1. Remove all back edges of D. Let D_A be the resulting acyclic digraph.

2. Assign a distinct positive label $x(r_i)$ to each dominator r_i, such that if r_i reaches r_j in D_A then $x(r_i) < x(r_j)$, $1 \leq i,j \leq m$ and $i \neq j$.

3. For each back edge (v,r_j) of D, include a new vertex w and an edge (v,w). Assign to w the negative label $-x(r_j)$.

4. Include the ordered sets

$$S=\{s_1,\dots,s_m\} \quad \text{and} \quad T=\{t_1,\dots,t_m\}$$

of new vertices. S and T are the sources and sinks of D^*, respectively. For $1 \leq j \leq m$, include the edge (s_j,r_j) and an edge to t_j from every vertex with label $-x(r_j)$. Assign to s_j and t_j the labels $x(r_j)$ and $-x(r_j)$, respectively.

5. Assign capacity 1 to each edge of D_A and infinite to those leaving S and entering T.

The sets $S=\{s_i\}$ and $T=\{t_i\}$ are in <u>normal order</u> if $x(s_i)<x(s_{i+1})$ and $x(t_i)>x(t_{i+1})$, $1 \leq i \leq m$, respectively.

<u>Lemma 2</u>: Let $S=\{s_1,\dots,s_m\}$ and $T=\{t_1,\dots,t_m\}$ be the sources and sinks of D^*, respectively in normal order. Then there is a one-to-one correspondence between cycles of D and s_j-t_j paths of D^*.

<u>Proof</u>: Let $v_1,\dots,v_k,v_1$ be a cycle C of D and v_1 its dominator. Then (v_k,v_1) is a back edge and there exists $w \in V(D^*)-V(D)$ such that $(v_k,w) \in E(D^*)$ and $x(w)=-x(v_1)$. Consequently, for some j, $1 \leq j \leq m$, $(s_j,v_1),(w,t_j) \in E(D^*)$ and therefore $s_j,v_1,\dots,v_k,w,t_j$ is a path in D^*. The converse is similar.

Lemma 3: Let $\{r_1, r_2\}$, $S=\{s_1, s_2\}$ and $T=\{t_1, t_2\}$ be the dominators, sources and sinks of D^* in normal order, respectively. Denote by f a S-T flow in D^* having value n and such that $f(s_1) \geq f(t_1)$. Then D has at least n edge disjoint cycles.

Proof: Since f has value n, D^* contains a set P, $|P|=n$, of S-T edge disjoint paths. Divide P into four subsets P_1, P_2, P_3 and P_4, consisting respectively of s_1-t_1, s_2-t_2, s_1-t_2 and s_2-t_1 paths. Clearly, $|P_3| = f(s_1)-|P_1|$ and $|P_4|=f(t_1)-|P_1|$. Hence $|P_3| \geq |P_4|$. We obtain the required cycles as follows. Each path of P_1 or P_2 corresponds to a cycle of D, according to lemma 2. Since D is reducible, each s_1-t_2 path contains r_2. Consequently, the union of a pair of cycles, one of P_3 and the other of P_4, contains two disjoint paths, of types s_1-t_1 and s_2-t_2, respectively, that is, two new cycles of D. In addition, the $|P_3|-|P_4|$ remaining s_1-t_2 paths can be transformed into an equal number of s_2-t_2 paths, by disregarding the s_1-r_2 subpaths and adding edges (s_2, r_2). A total of n edge disjoint cycles of D has been obtained▲

If $k>0$ is an integer denote by kD^* the network obtained from D as follows:

1. Construct D^*. Let $D_1^*, \ldots, D_k^*$ be k identical copies of D^*, with $S_i=\{s_{i1}, \ldots, s_{im}\}$ and $T_i=\{t_i, \ldots, t_{im}\}$, respectively the sets of sources and sinks of D_i^*, in normal order. The vertices of S_1 and T_k are the sources and sinks of kD^*, respectively.

2. For $1 \leq i \leq k$ and $1 \leq j \leq m$ include an edge $(t_{ij}, s_{i+1,j})$ and assign to it infinite capacity▲

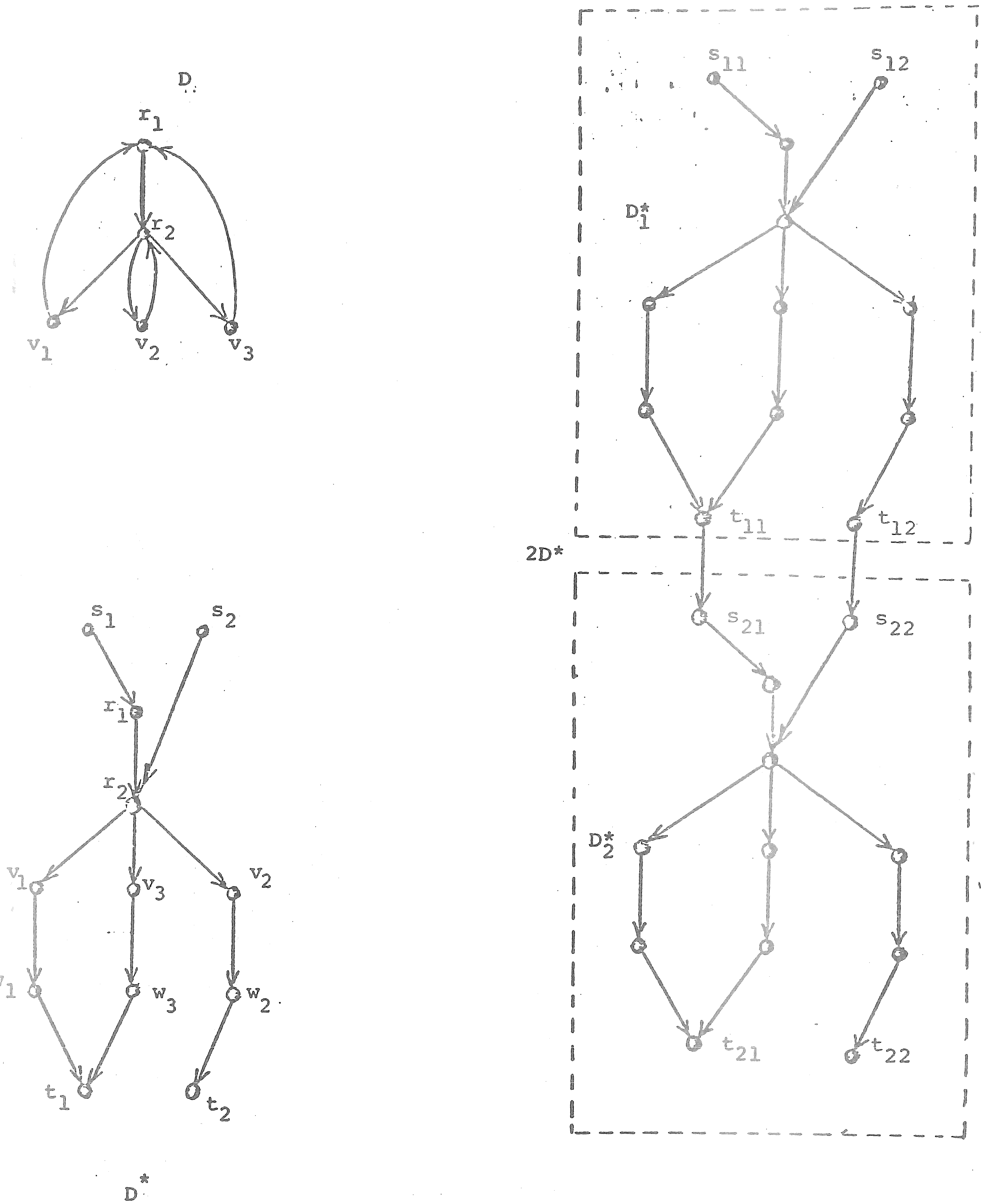

Figure 1 : A reducible digraph D and its associated networks.

Lemma 4: If kD^* has a cut X of capacity $n<\infty$ then D. has a feedback edge set β_E, such that $|\beta_E|\leq n$.

Proof: Let $Q=\{e \in E(D) \mid e \in X\}$. Then $|Q|\leq n$. Next, we show that Q is a feedback edge set. Suppose it is not. Then by lemma 2, D^* has a path p of the type s_j-t_j containing no edge of Q. By repeating p in each copy D^*_i of kD^* we obtain a S_1-T_k path in kD^* with no edge of X, a contradiction▲

Theorem 2: If $m\leq 2$, $\max|\alpha_E(D)|=\min|\beta_E(D)|$.

Proof: If m=1 the theorem follows from lemma 2 and the max-flow min-cut theorem [1] applied to D^*. When m=2, construct kD^*, $k=|E(D)|$. Let $S_1=\{s_{11},s_{12}\}$ and $T_k=\{t_{k1},t_{k2}\}$ be the sets of sources and sinks of kD^*, respectively in normal order, and f a maximum S_1-T_k flow in kD^*, having value n. Suppose, initially, $f(s_{i1}) < f(t_{i1})$, for all $1\leq i\leq k$. Since $f(s_{i1}) + f(s_{i2}) = f(t_{i1}) + f(t_{i2}) = n$, it follows that $f(s_{i2}) > f(t_{i2})$, $1\leq i\leq k$. Because $f(t_{i2}) = f(s_{i+1,2})$, $1\leq i<k$, we conclude that $f(s_{12}) > f(s_{22}) > \ldots > f(s_{k2})$. However, the latter inequality can not occur because $f(s_{12}) < |E(D)|$, $k=|E(D)|$ and all flow values $f(s_{i2})$ are non negative integers. Consequently, there exists some j, $1\leq j\leq k$, such that $f(s_{j1}) \geq f(t_{j1})$. Applying lemma 3 to D^*_j, we conclude that D contains at least n edge disjoint cycles, that is, $\max|\alpha_E(D)|\geq n$. By the max-flow min-cut theorem, kD^* has a cut of capacity n and using lemma 4 it follows that D contains a feedback edge set of size $\leq n$, that is, $\min|\beta_E(D)|\leq n$. Hence $\max|\alpha_E(D)| \geq \min|\beta_E(D)|$. Since $\max|\alpha_E(D)|\leq \min|\beta_E(D)|$ the equality follows▲

3. The Algorithms

The algorithms follow from the proof. Given the reducible digraph D, construct the network kD*, $k=|E(D)|$. Then find the minimum cut X of kD*. The edges of D which form X constitute the minimum cardinality feedback edge set β_E. For finding a maximum set of disjoint cycles α_E, let f be the maximum S-T flow in $|E(D)|D^*$. Next, identify the copy D_j^* in $|E(D)|D^*$ such that $f(s_{j1}) \geq f(t_{j1})$. Then use the construction of lemma 3 which transforms f into the desired α_E.

Both algorithms, for finding a minimum feedback edge set and a maximum set of edge disjoint cycles, have the same complexity as finding a maximum S-T flow in the network $|E(D)|D^*$, that is, polynomial in $|V(D)|$.

REFERENCES

[1] L.R.Ford and D.R. Fulkerson, Flows in Networks, Princeton University Press, Princeton, 1962.

[2] A.Frank and A.Gyárfás, Colloques Internationaux C.N.R.S. Problèmes Combinatoires et Théorie des Graphes (260), 157-158, 1976.

[3] M.Garey and D.S.Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H.Freeman, San Francisco, 1979.

[4] A.Gyárfás, Blocking the Loops of a Flow-Graph, Zeits. Angew. Math. Mech. 56 (1976), T330-331.

[5] M.S.Hecht and J.D.Ullman, Characterizations of Reducible Flow Graphs, J. of the ACM 21 (1974), 367-375.

[6] R.M.Karp, Reducibility among Combinatorial Problems, in R.E. Miller and J.W. Thatcher (eds.), Complexity of Computer Computations, Plenum Press, New York, 85-103, 1972.

[7] V.N.Kasyanov, Some Properties of Fully Reducible Graphs, Inf. Proc. Letters 2 (1973), 113-117.

[8] A.Shamir, A Linear Time Algorithm for Finding Minimum Cutsets in Reducible Graphs, SIAM J. Computing 8 (1979), 645-655.

[9] R.E.Tarjan, Testing Flow Graph Reducibility, J. Comp. Systems Sci. 9 (1974), 355-365.